

Problem list for Toan

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1 Set 1 - Deadline: Sun 25 Nov, 2007

1.1 Problems

- Problem 1

a) Prove that it is not possible to have these odd integer numbers $a_1, a_2, \dots, a_{2000}$ satisfying:

$$a_1^2 + \dots + a_{1999}^2 = a_{2000}^2$$

b) The product of 4 consecutive positive integers cannot be a square number

- Problem 2

Given 2031 arbitrary points **in** a circle. Prove that the circle could be divided into 3 parts such that the first part consists of 20 points, the second part consists of 11 points, and the last part has 2000 points.

1.2 Solutions

- Problem 1

a) Use the property that $a^2 \equiv 1[4]$ if a is odd

Note: this fact is fundamental and important to remember. Similar facts are $a^2 \equiv 0$ or $1[3]$, $a^2 \equiv 1[8]$ if a is odd

b) $P = n(n+1)(n+2)(n+3)$. Show that $a^2 < P < (a+1)^2$. Toan needs to figure out what is a ???

Note: this technique is commonly used to solve the problems involving square numbers. If we can only show $a^2 \leq P \leq (a+1)^2$ where $P = k^2$, we could then solve the following equations $k = a$, and $k = a + 1$ to prove or disprove the statement "P is a square number"

- Problem 2

Since the number of points on the circle is finite (*hau han*), the number of lines connecting any pairs of points is finite. Hence, the number of intersections (*giao diem*) of lines with the circle is finite. Thus, there exists a point A on the circle not lying (*nam tren*) any lines mentioned above.

Let B_1, \dots, B_{2301} be 2301 points in the circle. AB_i ($i = \overline{1..2301}$) cut the circle at C_1, \dots, C_{2301} counterclockwise (*theo chieu kim dong ho*).

Let D, E be points on the circle such that AD is in between AB_{20} and AB_{21} , while AE is in between AB_{31} and AB_{32} . It is obvious that the 3 arcs (*day cung*) AD, AE, DE divide the circle into 3 parts satisfying the problem requirement.

Note: The idea is simple. Just to get Toan's familiar with this type of problem. Notice the way they argue about how to choose point A.

2 Set 2 - Deadline: Sun 2 Dec, 2007

2.1 Problems

- Problem 1

Let $x, y, z \in \mathbb{R}$ satisfy $\begin{cases} x + y + z = 0 \\ -1 \leq x, y, z \leq 1 \end{cases}$
Prove that $x^2 + y^4 + z^6 \leq 2$

- Problem 2

Find all prime number p of the form $p = n^n + 1$ where $n \in \mathbb{N}^*$, knowing that p has the number of digits not greater than 19

- Problem 3

Solve the following equation system

$$\begin{cases} x^3(y^2 + 3y + 3) = 3y^2 \\ y^3(z^2 + 3z + 3) = 3z^2 \\ z^3(x^2 + 3x + 3) = 3x^2 \end{cases}$$

2.2 Solutions

- Problem 1

Firstly, we have $y^4 \leq y^2$ and $z^6 \leq z^2$ since $|y|, |z| \leq 1$, so $x^2 + y^4 + z^6 \leq x^2 + y^2 + z^2$. We will prove that $x^2 + y^2 + z^2 \leq 2$.

Since $x + y + z = 0$, there exists 2 numbers whose product is nonnegative. WLOG (without loss of generality - *ko mat tinh tong quat*), let x, y be the 2 numbers such that $xy \geq 0$.

We have $x^2 + y^2 + z^2 = (x + y)^2 - 2xy + z^2 = 2z^2 - 2xy \leq 2z^2 \leq 2$ (q.e.d)
(The equality occurs when, for example, $x = 0, y = -1, z = 1$)

Note: remember the fact that if $|x| \leq 1, x^n \leq x^m$ where $n \leq m$. So the number 2, 4, 6 in the problem could be replaced by any even number greater than 1.

Also, notice the way we pick up xy so that $xy \geq 0$. (Remind Anh Hai (AH) to search for a similar problem in AH's notes)

- Problem 2

(Idea: if n is odd (*le*), we could factor $p = n^n + 1 = (n + 1)[(n^{n-1} - n^{n-2}) + \dots + (n^2 - n) + 1] = (n + 1) \cdot A$. It is obvious that $A > 1$ if $n > 1$, which means p is no longer a prime number (contradicting). Therefore, $n = 2n_1$, and $p = (n^2)^{n_1} + 1$. Similarly, we could argue that n_1 is even (*chan*)....)

+ $n = 1 \Rightarrow p = 3$ (satisfying)

+ $n > 1$, let $n = 2^\alpha \cdot t$ where t is odd. Let $A = n^{2^\alpha}$, we have $p = A^t + 1$

If $t > 1 \Rightarrow 1 < (A + 1) < p$, and $(A + 1) | p$ (contradicting to p as a prime number) Thus, $t = 1$, and $n = 2^\alpha$

++ If $\alpha \geq 4, p \geq 16^{16} = 2^{64} = 16 \cdot (2^{10})^6 > 10 \cdot 10^{24} > 10 \cdot 10^{18} = 10^{19}$, contradicting to p has the number of digit not greater than 19 (*) ++ $\alpha = 1, 2, 3 \dots$

Note + The key thing here is to represent $n = 2^\alpha \cdot t$

+ Details in line (*) is not important. The important idea is to restrict $\alpha < 4$.

- Problem 3

It's easy to show that $t^2 + 3t + 3 > 0 \forall t$. Thus, $x, y, z \geq 0$, and we have

$$\begin{cases} x^3 = \frac{3y^2}{(y^2+3y+3)} \\ y^3 = \frac{3z^2}{(z^2+3z+3)} \\ z^3 = \frac{3x^2}{(x^2+3x+3)} \end{cases}$$

Let $f(t) = \frac{3t^2}{t^2+3t+3}$. Show that if $t_1 > t_2 \geq 0$, then $f(t_1) > f(t_2)$ (prove by yourself)

WLOG, let x be the maximal number among x, y, z . If $x > y \Rightarrow z^3 = f(x) > f(y) = x^3 \Rightarrow z > x$ (contradicting) $\Rightarrow x = y$ If $x > z \Rightarrow z^3 = f(x) >$

$$\begin{aligned}
 f(z) = y^3 = x^3 &\Rightarrow z > x \text{ (contradicting)} \Rightarrow x = z \text{ Thus, } x=y=z, \text{ and the} \\
 \text{equation system is reduced to } x^3 &= \frac{3x^2}{(x^2+3x+3)} \Rightarrow x^2(x(x^2 + 3x + 3) - 3) = 0 \Rightarrow \\
 x^2((x + 1)^3 - 4) &= 0 \\
 \Rightarrow x = 0, \text{ or } \sqrt[3]{4} - 1
 \end{aligned}$$

Note: the technique of constructing f , and showing that f is increasing (*dong bien*) or decreasing (*nglich bien*) is very useful in many problems, particularly in solving equation system.

3 Set 3 - Deadline: Sun 9 Dec, 2007

3.1 Problems

- Problem 1

Let X, Y, Z be three points on the BC, CA, AB sides of the triangle ABC (called in-triangle (*tam giac noi tiep*) of ABC

1) Let Y', Z' be the projected points of Y, Z on the side BC. Prove that if $\triangle XYZ \sim \triangle ABC$, then $Y'Z' = \frac{BC}{2}$

2) Among all triangles XYZ which are in-triangles of and similar to ABC, determine the one with smallest area

- Problem 2

a, b, c satisfy

$$\begin{cases} a + b + c = 0 \\ a^2 + b^2 + c^2 = 14 \end{cases}$$

Compute $P = 1 + a^4 + b^4 + c^4$

- Problem 3

Find all positive integer n such that $(n^2 + 9n - 2) \mid (n + 11)$

- Problem 4

Given a circle O and a point I in O. Construct through I 2 arcs MIN, EIF. M', N', E', F' are the midpoints of IM, IN, IE, and IF.

1) Prove that M'E'N'F' is an inscribed (*noi tiep*) quadrilateral (*tu giac*), and the radius of the circle circumscribing (*ngoai tiep*) M'E'N'F' is a constant

2) Suppose I is fixed. MIN, EIF vary, but always be perpendicular (*vuong goc*) to each other. Find the position of MIN, EIF such that M'E'N'F' has minimum area.

3.2 Solutions

- Problem 1

1) Let C' be the symmetric point of C over Y' . Prove that $\triangle BZC'$ and $\triangle CYC'$ are isosceles (*can*) $\Rightarrow Y'Z' = BC/2$

2) $\triangle XYZ \sim \triangle ABC \Rightarrow \frac{S_{XYZ}}{S_{ABC}} = (\frac{YZ}{BC})^2 \geq 1/4$ The equality occurs when X, Y, Z are the midpoints of BC, CA, AB respectively

Note:

+ If we could not solve 1), we could use the result of 1) to solve 2) very easy. This is to say that in the exam, try to read through the questions, and see if the latter could be solved using the result from the former or not

+ Remember triangle 1 is similar to triangle 2 at the ratio of k , then the area of triangle 1 is proportional to the area of triangle 2 at the ratio of k^2

- Problem 2

$$ab + bc + ca = -7 \Rightarrow a^2b^2 + b^2c^2 + c^2a^2 = 49 \Rightarrow P = 99$$

- Problem 3

$$(n+11) \mid (n^2+9n-2) = n(n+11)-2n-2 \Rightarrow (n+11) \mid (2n+2) = 2(n+11)-20 \Rightarrow (n+11) \mid 20 \Rightarrow n = 9$$

Note:

+ if we have expression1 \mid expression2, try to derive expression1 \mid constant (e.g. 20)

- Problem 4

1) Easy to show that $M'E'N'F'$ is an inscribed quadrilateral in circle

$$\triangle E'N'F' \sim \triangle ENF \Rightarrow R_{(E'N'F')} = 1/2R_{(ENF)}$$

2) Let P, Q be two points on MN, EF respectively s.t. $OP \perp MN, OQ \perp EIF$.

$$S_{M'N'E'F'} = 1/4S_{MNEF} = 1/8MN \cdot EF = 1/2MP \cdot EQ = 1/2 \cdot \sqrt{(R^2 - OP^2)(R^2 - OQ^2)} = 1/2 \cdot \sqrt{R^4 - R^2(OP^2 + OQ^2) + (OP \cdot OQ)^2} = 1/2 \cdot \sqrt{R^4 - R^2 \cdot OI^2 + (OP \cdot OQ)^2}$$

We only need to analyze $OP \cdot OQ$.

$$OP \cdot OQ \leq 1/2(OP^2 + OQ^2) = 1/2 \cdot OI^2 \Rightarrow S_{M'N'E'F'} \leq 1/4(2R^2 - OI^2).$$

The equality occurs when $OP = OQ$ or $\widehat{OIF} = 45^\circ$

$OP \cdot OQ \geq 0 \Rightarrow S_{M'N'E'F'} \geq 1/2 \cdot \sqrt{R^4 - R^2 \cdot OI^2}$. The equality occurs when $OP = 0$ (MN go through OI) or $OQ = 0$ (EF go through OI)

Note:

+ It is, in fact, a mistake when ask for **minimum** area instead of **maximum** area. Toan told that Toan could only solve for the **maximum** case, which is the correct question. However, it turns out that the **minimum** case is even easier :D

4 Set 4 - Deadline: Sun 25 Dec, 2007

4.1 Problems

- Problem 1

Let $x, y > 0$ s.t. $x + y = 1$. Find the minimal value of the following expression:

$$P = \left(x^2 + \frac{1}{y^2}\right)\left(y^2 + \frac{1}{x^2}\right)$$

- Problem 2

Given a circle (O, R) , let A, B be two fixed points on the circle and $AB = R\sqrt{3}$.

1) Suppose M is a varying point on the larger arc AB of the circle. The incircle of MAB touches (is tangent to) MA and MB at E and F respectively. Prove that EF is always tangent to a fixed circle when M varies

2) Find the collection of P such that the line (d) perpendicular to $(vuong\ goc)$ OP at P cuts the segment AB

- Problem 3

Give a circle O of radius 1 and 8 arbitrary points inside the circle (including the boundary). Prove that there exists 2 points such that the distance between them is less than 1

- Problem 4

Prove the following inequalities where a, b, c are non-negative integers:

$$3 \leq \frac{1 + \sqrt{a}}{1 + \sqrt{b}} + \frac{1 + \sqrt{b}}{1 + \sqrt{c}} + \frac{1 + \sqrt{c}}{1 + \sqrt{a}} \leq 3 + a + b + c$$

4.2 Solutions

- Problem 1

$$P = (x^2 + \frac{1}{y^2})(y^2 + \frac{1}{x^2}) = x^2y^2 + \frac{1}{x^2y^2} + 2$$

$$0 < xy \leq \frac{(x+y)^2}{4} = \frac{1}{4}$$

The problem is equivalent (*tuong duong*) to find the minimum of

$$P = t^2 + \frac{1}{t^2} + 2$$

where $0 < t \leq 1/4$

Solution 1

$$x^2y^2 + \frac{1}{256x^2y^2} \geq 1/8$$

$$\frac{255}{256x^2y^2} \geq \frac{255}{256(1/4)^2} \geq 255/16$$

Thus, $P \geq 289/16$. Equality occurs when $x = y = 1/2$.

Solution 2

Consider the function

$$f(t) = t + 1/t$$

where t in $(0, 1)$. We prove that $f(t)$ is an decreasing function. Consider $0 < t_1 < t_2 < 1$, we have

$$f(t_1) - f(t_2) = (t_1 - t_2)(1 - 1/(t_1t_2)) > 0$$

Thus, $f(t_1) > f(t_2)$, or f is decreasing.

Return back to our problem we have $x^2y^2 \leq 1/16$, so $f(x^2y^2) \geq f(1/16) = 257/16$

Thus, $P \geq 289/16$. Equality occurs when $x = y = 1/2$.

Note:

1) The reason why the original problem is equivalent with the problem of the new variable t is:

For any particular values of $x, y > 0$ s.t $x + y = 1$, there always exist a value t in $(0, 1/4]$ s.t $t = xy$ as $0 < xy \leq 1/4$.

On the other hand, for any value t in $(0, 1/4]$, there always exists values of x, y s.t

$$\begin{cases} x + y = 1 \\ xy = t \end{cases}$$

This is the problem of *bien luan phuong trinh*. x, y are the roots of the equation $X^2 - X + t = 0$.

Since $\Delta = 1 - 4t \geq 0$, the equation has two roots. Since the sum, and product of the two roots > 0 , the two roots are > 0 .

2) Think about the reason why in solution 1, we know to use $1/256$

3) Try to familiarize yourself with the techniques of decreasing (or increasing) functions in solution 2

• Problem 2

1)

$$\widehat{AOB} = 120^\circ \Rightarrow \widehat{AMB} = 60^\circ$$

$$\Rightarrow \triangle MEF$$

is equilateral (*deu*).

Let I be the midpoint of AB , and A', B', I' be the projection of A, B, I on EF respectively.

$$AA' = \sqrt{3}/2AE, BB' = \sqrt{3}/2BF$$

$$\Rightarrow AA' + BB' = \sqrt{3}/2(AE + BF) = \sqrt{3}/2AB = 3/2R$$

$$\Rightarrow II' = 3/4R = \text{const}$$

Thus, EF is tangential to the fixed circle $(I, 3/4R)$.

2) To imagine how the set of P is defined, draw a line d' passing through O . A_1, B_1 be the projected points of A, B on d_1 . Any point P in the segment $A'B'$ will satisfy the condition.

Similarly, draw another line d_2 and points A_2, B_2 .

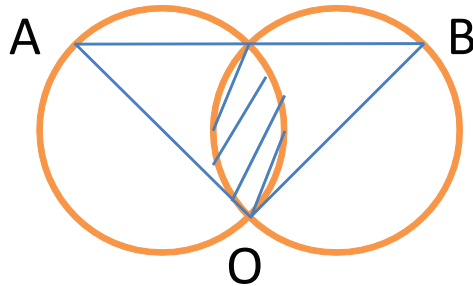
Similarly, draw another line d_3 and points A_3, B_3 .

What are the relationships of A_1, A_2, A_3 , and B_1, B_2, B_3 ?

A_1, A_2, A_3 lie on the circle C_1 of diameter (*duong kinh*) OA

B_1, B_2, B_3 lie on the circle C_2 of diameter (*duong kinh*) OB

The collection of P is all points in the circles C_1 and C_2 except the intersection (see the figure, and think about why the collection is so)



Note:

For 1) generalize for $AB = d = \text{constant}$

• Problem 3

There always exist 7 points different from the center O , called A_1, A_2, \dots, A_7 .

Also, there always exist 2 points A_i, A_j such that $\widehat{A_i O A_j} \leq 360^\circ/7 < 60^\circ$

Consider $\triangle O A_i A_j$, since $\widehat{A_i O A_j} < 60^\circ$, $A_i A_j < \max\{A_i O, A_j O\} \leq 1$

• Problem 4

$$P = \frac{1 + \sqrt{a}}{1 + \sqrt{b}} + \frac{1 + \sqrt{b}}{1 + \sqrt{c}} + \frac{1 + \sqrt{c}}{1 + \sqrt{a}}$$

+ Use Cauchy inequality, we have $P \geq 3$

+ $\frac{1 + \sqrt{a}}{1 + \sqrt{b}} \leq 1 + \sqrt{a} \leq 1 + a \Rightarrow (q.e.d)$

5 Set 5 - Deadline: Sun 9 Jan, 2008

5.1 Problems

- Problem 1

Any triangle with two angle bisectors (*phan giac*) of equal lengths is isosceles (*can*).

- Problem 2

Prove that among 7 arbitrary natural numbers, we could always pick 4 numbers such that their sum is divisible by 4.

- Problem 3

Given a triangle ABC. On side BC, take 2 points M, N such that $\widehat{BAM} = \widehat{CAN}$. Prove that:

- $\frac{BM}{CN} \cdot \frac{CM}{BN} = \left(\frac{AM}{AN}\right)^2$
- $\frac{BM}{BN} \cdot \frac{CM}{CN} = \left(\frac{AB}{AC}\right)^2$
- $\frac{BM}{CN} + \frac{CM}{BN} \geq 2\frac{AM}{AN}$

- Problem 4

Given 6 real numbers in the range $[0; 1]$. Show that:

$$(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5)(x_5 - x_6)(x_6 - x_1) \leq \frac{1}{16}$$

5.2 Solutions

- Problem 1

This is the Steiner-Lemus theorem.

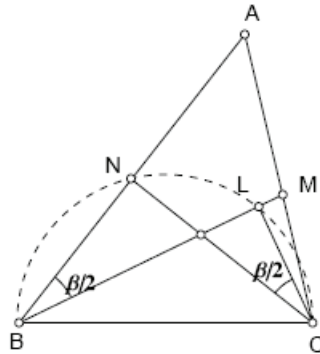
<http://mathworld.wolfram.com/Steiner-LehmusTheorem.html>

http://en.wikipedia.org/wiki/Steiner-Lehmus_theorem

<http://forumgeom.fau.edu/FG2005volume5/FG200525.pdf>

First proof:

Suppose $\beta < \gamma$ in triangle ABC . We show that the bisector BM is longer than the bisector CN .



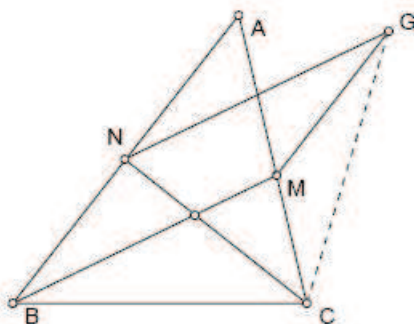
Choose a point L on BM such that $\angle NCL = \frac{1}{2}\beta$. Then B, N, L, C are concyclic since $\angle NBL = \angle NCL$. Note that

$$\angle NBC = \beta < \frac{1}{2}(\beta + \gamma) = \angle LCB,$$

and both are acute angles. Since smaller chords of a circle subtend smaller acute angles, we have $CN < BL$. It follows that $CN < BM$.

Second proof:

Suppose the bisectors BM and CN in triangle ABC are equal. We shall show that $\beta = \gamma$. If not, assume $\beta < \gamma$. Compare the triangles CBM and BCN . These have two pairs of equal sides with included angles $\angle CBM = \frac{1}{2}\beta < \frac{1}{2}\gamma = \angle BCN$, both of which are acute. Their opposite sides therefore satisfy the relation $CM < BN$.



Complete the parallelogram $BMGN$, and consider the triangle CNG . This is isosceles since $CN = BM = NG$. Note that

$$\angle CGN = \frac{1}{2}\beta + \angle CGM,$$

$$\angle GCN = \frac{1}{2}\gamma + \angle GCM.$$

Since $\beta < \gamma$, we conclude that $\angle CGM > \angle GCM$. From this, $CM > GM = BN$. This contradicts the relation $CM < BN$ obtained above.

• Problem 2

Lemma: for any 3 natural numbers, there exist 2 numbers such that their sum is divisible by 2.

Thus, WLOG:

For a_1, a_2, a_3 , we have $a_1 + a_2 = 2k_1 (k_1 \in \mathbb{Z})$

For a_3, a_4, a_5 , we have $a_3 + a_4 = 2k_2 (k_2 \in \mathbb{Z})$

For a_5, a_6, a_7 , we have $a_5 + a_6 = 2k_3 (k_3 \in \mathbb{Z})$

For k_1, k_2, k_3 , we have $k_1 + k_2 \equiv 2$

Thus, we have $a_1 + a_2 + a_3 + a_4$ such that they are divisible by 4

Note:

It may be tricky to find a nice way to solve this problem, but it is very to find

an ugly way to solve the problem by considering many cases. Thus, Toan is expected to solve this problem no matter how nice the solution is.

• Problem 3

Lemma: for any triangle ABC, its area could be computed through the formula

$$S = 1/2 AB \cdot AC \cdot \sin A$$

a)

$$\frac{BM}{CN} \cdot \frac{CM}{BN} = \frac{S_{ABM}}{S_{ACN}} \cdot \frac{S_{ACM}}{S_{ABN}} = \frac{AB \cdot AM \cdot \sin(\widehat{BAM})}{AC \cdot AN \cdot \sin(\widehat{CAN})} \cdot \frac{AC \cdot AM \cdot \sin(\widehat{CAM})}{AB \cdot AN \cdot \sin(\widehat{BAN})} = \left(\frac{AM}{AN}\right)^2$$

b) Similar to a)

c) Cauchy

• Problem 4

Let $y_k = x_k - x_{k+1}, k = \overline{1,6}$ where $x_7 \equiv x_1$. We have $y_i \in [-1, 1]$, and $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 0$. The problem is to show that:

$$A = y_1 \cdot y_2 \cdot y_3 \cdot y_4 \cdot y_5 \cdot y_6 \leq 1/16$$

- + If there exists $y_i = 0 \Rightarrow A = 0 < 1/16$ (q.e.d)
- + If the number of negative numbers is odd (*le*) $\Rightarrow A < 0 < 1/16$
- + Since $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 0$, there exist at least one negative and one positive number. Thus, the number of negative numbers is either 2, or 4.
- + If there are 2 negative numbers, WLOG, let them be y_1, y_2 . We have,

$$A \leq |y_1| |y_2| \left(\frac{y_3 + y_4 + y_5 + y_6}{4}\right)^4 \leq 1.1 \cdot (2/4)^4 = 1/16$$

The equality occurs when $y_3 = y_4 = y_5 = y_6 = 1/2, y_1 = -1, y_2 = -1 \Rightarrow x_i$

+ If there are 4 negative numbers, WLOG, let the 2 positive numbers be y_1, y_2 . We have,

$$A \leq y_1 \cdot y_2 \left(\frac{|y_3| + |y_4| + |y_5| + |y_6|}{4}\right)^4 \leq 1.1 \cdot (2/4)^4 = 1/16$$

The equality occurs when $y_3 = y_4 = y_5 = y_6 = -1/2, y_1 = 1, y_2 = 1 \Rightarrow x_i$

Note:

+ “Changing variable lesson”: the problem looks scary at first, by changing the variable, it looks clearer and easier to tackle.

+ Learn a lesson of considering different cases with a clear case division, and making WLOG assumption to reduce the number of cases.

6 Set 6 - Deadline: Sun 27 Jan, 2008

6.1 Problems

- Problem 1

1) Solve the equation

$$\sqrt{x(x+1)} + \sqrt{x(x+2)} = \sqrt{x(x-3)}$$

2) Find a so that the following equation has only one root

$$\frac{x^2 - (3a - 2)x + 2a^2 - 5a - 3}{x^2 + 5x - 14} = 0$$

- Problem 2

With $x, y, z > 0$. Show that

$$\frac{2x}{x^6 + y^4} + \frac{2y}{y^6 + z^4} + \frac{2z}{z^6 + x^4} \leq \frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4}$$

- Problem 3

In the coordinator system xOy , given $A(-3, 0), B(-1, 0)$. Consider 2 points M, and N varies on the vertical (*tung, du*'ng*)axis such that AM is perpendicular (*vuong goc*) to BN.

1) Prove that the circle with diameter (*duong kinh*) MN always go through 2 fixed points, and find the coordinators (*toa do*) of them.L

2) Find the collection of those points which are the centers of the circles circumscribe $\triangle AMN$. Determine M, N so that $\triangle AMN$ has minimum area

6.2 Solutions

- Problem 1

$$\text{Conditions } \begin{cases} x \leq -2 \\ x = 0 \\ x \geq 3 \end{cases}$$

$$x = 0 \text{ and } x = -\sqrt{\frac{28}{3}}$$

Note:

+ Be careful in determining the value range of x , and not to jump to the conclusion that $x > 0$

- Problem 2

$$\text{Conditions } \begin{cases} x \neq 2 \\ x \neq -7 \end{cases} \quad (1)$$

Discriminant $\Delta = (a + 4)^2$. For the equation to have only one solution, we have two cases:

+ The equation has two **coincident** (*trùng = kép*) roots satisfying (1) \Leftrightarrow

$$\begin{cases} \Delta = 0 \\ x_0 = \frac{3a-2}{2} \neq 2, -7 \end{cases} \Leftrightarrow \begin{cases} a = -4 \\ a \neq 2 \\ a \neq -4 \end{cases} \quad (\text{contradiction})$$

+ The equation has two **distinct** (*phân biệt*) roots ($x_1 = 2a + 1$ and $x_2 = a - 3$) satisfying one of them is equal to 2 (or -7) while the other is different from 2, and -7.

$$++ \Delta > 0 \Rightarrow a \neq -4 \quad (2)$$

$$++ \text{ If } x_1 = 2 \Rightarrow a = 1/2 \text{ (satisfying as } x_2 = 7/2)$$

$$++ \text{ If } x_1 = -7 \Rightarrow a = -4 \text{ (contradicting to (2))}$$

$$++ \text{ If } x_2 = 2 \Rightarrow a = 5 \text{ (satisfying as } x_1 = 11)$$

$$++ \text{ If } x_2 = -7 \Rightarrow a = -4 \text{ (contradicting to (2))}$$

Thus $a = 1/2$ or 5.

- Problem 3

$$1) H(-\sqrt{3}, 0) \text{ and } H'(\sqrt{3}, 0)$$

2) The circumcenter of triangle AMN lie on the perpendicular bisector (*trung trục*) of [AT].

$$S_{AMN} \geq 3\sqrt{3}$$

7 Set 7 - Deadline: Friday, 15 Feb, 2008

7.1 Problems

- Problem 1

1) Solve and argue on a: $(x^2 - 5x + 6)\sqrt{(x^2 - 5ax + 6a^2)} = 0$

2) For what values of a, the following equation system has at least 1 root (x, y) satisfying $x, y > 0$. Find all roots for each value of a.

$$\begin{cases} x + y + \frac{1}{x} + \frac{1}{y} = 4 \\ x^2 + y^2 + \frac{1}{x^2} + \frac{1}{y^2} = \sqrt{2 - a^2} + \sqrt{2 - \frac{1}{a^2}} + \frac{a^2 + 1}{a} \end{cases}$$

- Problem 2

1) Find all positive integers x, y such that $\begin{cases} 2^x = 2y \\ 2^y = 2x \end{cases}$

2) P(x) is a polynomial of degree 3 with the coefficient of x^3 is an integer $\neq 0, -1$. Knowing that $P(1999) = 2000$ and $P(2000) = 2001$.

Show that $P(2001) - P(1998)$ is a non-prime number.

- Problem 3

Given $x_1, x_2, x_3, x_4 > 0$ such that $\sum_{i=1}^4 x_i = 1$. Find the minimum value of

$$T = \frac{\sum_{i=1}^4 x_i^4}{\sum_{i=1}^4 x_i^3}.$$

- Problem 4

Given the triangle ABC with all edges of different lengths. G is the center of $\triangle ABC$. A_1, B_1, C_1 are respectively the symmetric points of A, B, C through G. Knowing that $AB = 2 \cdot BC$, and $S_{A_1 B_1 C_1} = 72$.

Compute the area of the common hexagon of $\triangle ABC$ $\triangle A_1 B_1 C_1$.

7.2 Solutions

- Problem 1

1) Argue a on those intervals divided by $2/3$, 1 , and $3/2$. For each interval, argue how many distinct roots the equation has.

2) $x = y = 1$. It remains to work on $4 = \sqrt{2 - a^2} + \sqrt{2 - \frac{1}{a^2}} + \frac{a^2 + 1}{a}$.

Use Bunhiacopski \Rightarrow Only with $a = 1$, the equation system has $x = y = 1$.

- Problem 2

1) First, show that $x = y$.

Thus, the equation system become $2^x = 2x$. Since x is a positive integer, it's easy to show by induction (*quy nap*) that $2^x > 2x$ for all $x \geq 3$. Therefore, $x = y = 1, 2$.

2) Consider $Q(x) = P(x) - (x+1)$. $Q(x)$ has two roots $x = 1999, 2000$. Since $Q(x)$ is a polynomial of degree 3 with the highest coefficient $a \in \mathbb{Z}, a \neq 0, -1$, we have:

$$Q(x) = (x - 1999)(x - 2000)(ax + b)$$

$$\Rightarrow P(2001) = 2002 + 2(2001a + b), \text{ and } P(1998) = 1999 + 2(1998a + b)$$

$$\Rightarrow P(2001) - P(1998) = 3 + 2.3a = 3.(2a + 1)$$

$$3 \mid (P(2001) - P(1998)), \text{ and } (P(2001) - P(1998)) \text{ is different from } 3 \text{ (} a \neq 0).$$

Therefore, $P(2001) - P(1998)$ is a non-prime number.

Note:

* Looks like we don't need the condition that $a \neq -1$.

- Problem 3

WLOG (*without loss of generality*), we may assume that $x_1 \leq x_2 \leq x_3 \leq x_4 \Rightarrow x_1^3 \leq x_2^3 \leq x_3^3 \leq x_4^3$. Using Tchebychev inequality, we obtain:

$$\frac{\sum_{i=1}^4 x_i}{4} \cdot \frac{\sum_{i=1}^4 x_i^3}{4} \leq \frac{\sum_{i=1}^4 x_i^4}{4}$$

Since $\sum_{i=1}^4 x_i = 1$, we derive that

$$\frac{\sum_{i=1}^4 x_i^4}{\sum_{i=1}^4 x_i^3} \geq 1/4$$

The equality occurs when $x_i = 1/4$ for all i .

Note:

Learn a bit about Tchebychev inequality.

- Problem 4

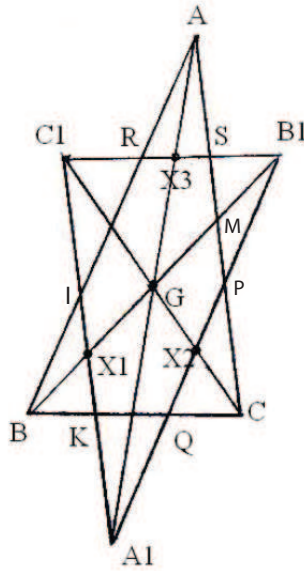
* First, show that $X_1B = 1/4BB_1$, $X_2C = 1/4CC_1$, and $X_3A = 1/4AA_1$

* $\frac{S_{B_1SP}}{S_{B_1C_1A_1}} = \frac{B_1M}{B_1X} = 1/9$. Similarly, we have

$$\frac{S_{C_1RI}}{S_{C_1A_1B_1}} = 1/9$$

$$\frac{S_{A_1QK}}{S_{A_1B_1C_1}} = 1/9$$

Thus $S_{RIKQPS} = S_{A_1B_1C_1} - S_{A_1QK} - S_{C_1RI} - S_{B_1SP} = 2/3S_{A_1B_1C_1} = 2/3 * 72 = 48$ (area units)



Note:

Don't know where $AB = 2BC$ is used??

8 Set 8 - Deadline: Friday, 10 March, 2008

8.1 Problems

- Problem 1

Prove that for all n as a positive integer, we have: $5^n \cdot (5^n + 1) - 6^n \cdot (3^n + 2^n) \equiv 91$

- Problem 2

Given $x, y > 0$ such that $x \cdot y = 1$. Find the maximum values of:

$$A = \frac{x}{x^4 + y^2} + \frac{y}{y^4 + x^2}$$

- Problem 3

Solve the equation: $\sqrt{x+1} + 2(x+1) = x-1 + \sqrt{1-x} + 3\sqrt{1-x^2}$

8.2 Solutions

- Problem 1

Prove that the expression is divisible by 7, and 13

- Problem 2

Apply Cauchy inequality for the denominators (*mau so*), we have:

$$A \leq \frac{x}{2x^2y} + \frac{y}{2y^2x} = \frac{1}{xy} = 1$$

Note:

+ The problem is easy, but might look intimidating at first. When first look at it, people might be tempted to use complicated transformation or inequalities. However, it turns out that only simple Cauchy inequality for denominators will do. **Lesson** learned is must try out to see first, don't just stare at the problem.

+ Another thought for the problem is we observe that the inequality is not so balanced in terms of degree e.g. $x^4 + y^2$. So we add things to make it balance e.g. $x^4 + y^2 = x^4 + y^2xy$, and it turns out that A could be simplified as

$$A = \frac{x}{x^4 + y^2} + \frac{y}{y^4 + x^2} = \frac{x}{x^4 + y^2xy} + \frac{y}{y^4 + x^2xy} = \frac{1}{x^3 + y^3} + \frac{1}{y^3 + x^3} = \frac{2}{y^3 + x^3}$$

Now, Cauchy inequality will come naturally.

- Problem 3

Condition: $|x| \leq 1$

Let $a = \sqrt{1+x}$, and $b = \sqrt{1-x}$. The equation is now equivalent to:

$$a^2 + b^2 = 2 \tag{1}$$

$$a + 2a^2 = -b^2 + b + 3ab \tag{2}$$

$$\begin{aligned} a + 2a^2 &= -b^2 + b + 3ab \\ \Rightarrow a - b + 2a(a - b) + b(b - a) &= 0 \\ \Rightarrow (a - b)(1 + 2a + b) &= 0 \end{aligned} \tag{3}$$

From (1), and (3), it is easy to derive that $x = 0$, or $x = -24/25$.

Note:

Notice the technique of changing variables, otherwise it's very hard to perform factoring.